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Elements of Ordinary Differential Initial Value Problem Approximation

1.1 INTRODUCTION

To obtain accurate numerical solutions to differential equations governing physical systems has always been an important problem with scientists and engineers. These differential equations basically fall into two classes, *ordinary* and *partial*, depending on the number of independent variables present in the differential equations: one for ordinary and more than one for partial.

The general form of the ordinary differential equation can be written as

$$L[y] = r \tag{1.1}$$

where L is a *differential operator* and r is a given function of the independent variable t. The order of the differential equation is the order of its highest derivative and its degree is the degree of the derivative of the highest order after the equation has been rationalized. If no product of the dependent variable $y(t)$ with itself or any of its derivatives occur, the equation is said to be linear, otherwise it is nonlinear. A linear differential equation of order m can be expressed in the form

$$L[y] = \sum_{p=0}^{m} f_p(t)\, y^{(p)}(t) = r(t) \tag{1.2}$$

in which $f_p(t)$ are known functions. The general nonlinear differential equation of order m can be written as

$$F(t, y, y', \ldots, y^{(m-1)}, y^{(m)}) = 0 \tag{1.3}$$

or

$$y^{(m)}(t) = f(t, y, y', \ldots, y^{(m-1)}) \tag{1.4}$$

which is called a canonical representation of differential equation (1.3). In such a form, the highest order derivative is expressed in terms of the lower order derivatives and the independent variable. The general solution of the mth order ordinary differential equation contains m independent arbitrary constants. In order to determine the arbitrary constants in the general solution if the m conditions are prescribed at one point, these are called *initial conditions*. The differential equation together with the initial conditions is called the *initial value problem*. Thus, the mth order initial value problem can be expressed as

$$y^{(m)}(t) = f(t, y, y', \ldots, y^{(m-1)})$$
$$y^{(p)}(t_0) = y_0^{(p)}, \; p = 0, 1, 2, \ldots, m-1 \tag{1.5}$$

If the m conditions are prescribed at more than one point, these are called *boundary conditions*. The differential equation together with the boundary conditions is known as the ***boundary value problem.*** We shall now discuss the basic concepts needed for the solution of initial value problems.

1.2 INITIAL VALUE PROBLEMS

The mth order initial value problem of Equation (1.5) is equivalent to the following system of m first order equations:

$$\begin{aligned} y' = v_1' &= v_2 & v_1(t_0) &= y_0 \\ v_2' &= v_3 & v_2(t_0) &= y_0' \\ &\vdots & &\vdots \\ v_{m-1}' &= v_m & v_{m-1}(t_0) &= y_0^{(m-2)} \\ v_m' &= f(t, v_1, v_2, \ldots, v_m) & v_m(t_0) &= y_0^{(m-1)} \end{aligned} \tag{1.6}$$

In vector notations it can be written as

$$\frac{d\mathbf{v}}{dt} = \mathbf{f}(t, \mathbf{v}), \; \mathbf{v}(t_0) = \boldsymbol{\eta} \tag{1.7}$$

where

$$\mathbf{v} = [v_1 \; v_2 \; \ldots \; v_m]^T,$$
$$\mathbf{f}(t, \mathbf{v}) = [v_2 \; v_3 \; \ldots \; f(t, v_1, v_2, \ldots, v_m)]^T,$$
$$\boldsymbol{\eta} = [y_0 \; y_0' \ldots \; y_0^{(m-1)}]^T$$

We shall, therefore, be concerned with methods for finding out numerical approximations to the solution of the equation

$$\frac{dy}{dt} = f(t, y), \; y(t_0) = y_0 \tag{1.8}$$

However, before attempting to approximate the solution numerically, we must ask if the problem has any solution. This can be answered easily in the case of initial value problem for ordinary differential equation by Theorem 1.1.

THEOREM 1.1 *We assume that $f(t, y)$ satisfies the following conditions:*

(i) *$f(t, y)$ is a real function,*

(ii) *$f(t, y)$ is defined and continuous in the strip*

$$t \in [t_0, b], \; y \in (-\infty, \infty),$$

(iii) *there exists a constant L such that for any $t \in [t_0, b]$ and for any two numbers y_1 and y_2*

$$|f(t, y_1)-f(t, y_2)| \leq L\,|y_1-y_2|,$$

where L is called the Lipschitz constant.

Then, for any y_0 the initial value problem (1.8) *has a unique solution $y(t)$ for $t \in [t_0, b]$.*

We will always assume the existence and uniqueness of the solution and also that $f(t, y)$ has continuous partial derivatives with respect to t and y of as high an order as we desire.

1.3 DIFFERENCE EQUATIONS

In order to develop approximations to differential equations, we define the following operators:

$Ey(t) = y(t+h)$	The shift operator
$\Delta y(t) = y(t+h)-y(t)$	The forward-difference operator
$\nabla y(t) = y(t)-y(t-h)$	The backward-difference operator
$\delta y(t) = y\left(t+\frac{h}{2}\right)-y\left(t-\frac{h}{2}\right)$	The central-difference operator
$\mu y(t) = \frac{1}{2}\left[y\left(t+\frac{h}{2}\right)+y\left(t-\frac{h}{2}\right)\right]$	The average operator
$Dy(t) = y'(t)$	The differential operator

where h is the difference interval.

Repeated applications of the difference operators lead to the following higher order differences:

$$\Delta^n y(t) = \sum_{k=0}^{n} (-1)^k \frac{n!}{k!\,(n-k)!}\, y(t+(n-k)h) \tag{1.9}$$

$$\nabla^n y(t) = \sum_{k=0}^{n} (-1)^k \frac{n!}{k!\,(n-k)!}\, y(t-kh) \tag{1.10}$$

$$\delta^{2n} y(t) = \sum_{k=0}^{2n} (-1)^k \frac{(2n)!}{k!\,(2n-k)!}\, y(t+(n-k)h) \tag{1.11}$$

For linking the difference operators with the differential operator we consider Taylor's formula

$$y(t+h) = y(t)+hy'(t)+\frac{h^2}{2!}y''(t)+\dots \tag{1.12}$$

In operator notations we can write

$$Ey(t) = \left(1+hD+\frac{(hD)^2}{2!}+\dots\right)y(t)$$

A kth order linear inhomogeneous difference equation with constant coefficients is of the form

$$a_0 y_{n+k} + a_1 y_{n+k-1} + \dots + a_k y_n = g_n \tag{1.14}$$

where $a_j, j = 0, 1, \dots, k$, are constants independent of n, and $a_0 \neq 0$, $a_k \neq 0$.

The solution y_n of Equation (1.14) consists of a solution to the homogeneous equation, say $y_n^{(H)}$, and a particular solution, say $y_n^{(P)}$ of the inhomogeneous part.

Substituting $g_n = 0$ in (1.14), we get the homogeneous difference equation

$$a_0 y_{n+k} + a_1 y_{n+k-1} + \dots + a_k y_n = 0 \tag{1.15}$$

To find the solution of (1.15), we use the trial solution

$$y_n = A\xi^n \tag{1.16}$$

where $A \neq 0$ is a constant and ξ is a number to be determined.

Inserting (1.16) in (1.15), we find that nontrivial solutions exist if ξ is a root of the polynomial

$$a_0 \xi^k + a_1 \xi^{k-1} + \dots + a_k = 0 \tag{1.17}$$

This equation is called the *characteristic equation* of the difference equation (1.15).

Thus, if ξ_j are the distinct roots of (1.17), then we may write

$$y_n^{(H)} = \sum_{j=1}^{k} b_j \xi_j^n = \sum_{j=1}^{k} b_j \exp(n \log \xi_j) \tag{1.18}$$

where b_j are the arbitrary constants.

Let us assume now that $\xi_1 (= \xi_2)$ is a double root of (1.17), and that all other roots ξ_j, $j = 3, 4, \dots, k$, are distinct. Then we would get $k-1$ solutions of the form $\xi_1^n, \xi_3^n, \dots, \xi_k^n$. However, it can easily be verified by substitution that if ξ_1 is a double root, then $n\xi_1^n$ is also a solution of (1.15). Thus the general solution of (1.15) becomes

$$y_n^{(H)} = b_1 \xi_1^n + b_2 n \xi_1^n + \sum_{j=3}^{k} b_j \xi_j^n \tag{1.19}$$

In general, if the characteristic Equation (1.17) has roots $\xi_j, j = 1, 2, \dots, p$, and the roots ξ_j has multiplicity γ_j, where $\sum_{j=1}^{p} \gamma_j = k$, ther the general solution of (1.15) is given by

$$\begin{aligned} y_n^{(H)} = {} & [b_{11} + b_{12} n + b_{13} n(n-1) + \dots + b_{1\gamma_1} n(n-1) \dots (n-\gamma_1+2)]\xi_1^n \\ & + [b_{21} + b_{22} n + b_{23} n(n-1) + \dots + b_{2\gamma_2} n(n-1) \dots \\ & \dots (n-\gamma_2+2)]\xi_2^n \\ & \dots + [b_{p1} + b_{p2} n + b_{p3} n(n-1) + \dots + b_{p\gamma_p} n(n-1) \dots \\ & \dots (n-\gamma_p+2)]\xi_p^n \end{aligned} \tag{1.20}$$